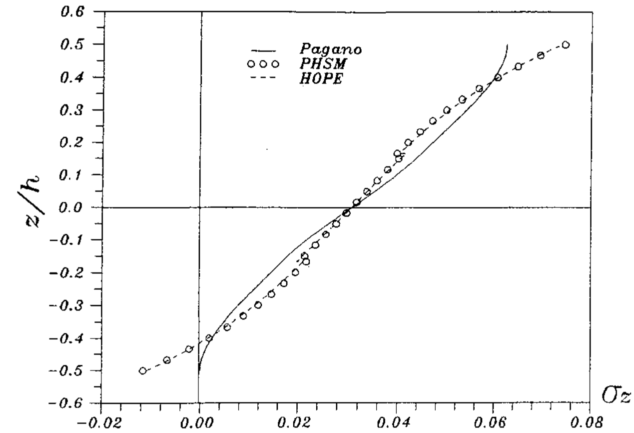

Fig. 3 Distribution of $\bar{\sigma}_z$ $(a/2,\ a/2,\ z/h)$ for (0/90/0 deg) $a/h=4$ plate.

composite plates are subjected to sinusoidal loading. Both examples are simply supported and of the same thickness per ply. Various span-to-thickness (a/h) ratios are investigated. The full plate is divided into 2, 6, and 10 strips, and one term of $Y_m(y)$ series is used because of only one term in sinusoidal loading.

The displacement analysis of the present model is more accurate than those of Kant and Pandya[5] and the refined C^1 higher-order plate theory of Reddy[6] in these examples. It is thought to be attributed to the overcome interlaminar traction discontinuity.

Both the present PHSM and the higher-order plate element[5] (HOPE) closely predict the through thickness variation of in-plane deformation $\bar{u}$ and the flexural stresses $(\bar{\sigma}_x,\ \bar{\sigma}_y,\ \bar{\tau}_{xy},\ \bar{\sigma}_z)$. The $\bar{\tau}_{xz}$, $\bar{\tau}_{yz}$, $\bar{\sigma}_z$ for $a/h=4$ (0/90/0 deg) plates are shown in Figs. 1-3. The PHSM can yield a rational distribution of $(\bar{\tau}_{xz},\ \bar{\tau}_{yz})$ satisfying traction boundary conditions. The $\bar{\tau}_{yz}$ is in excellent agreement with the exact solution[4]. There exists deviation of $\bar{\tau}_{xz}$, however, it is continuous through thickness. On the other hand, obvious discontinuity is observed in HOPE. The continuity of $\bar{\sigma}_z$ is neglected since the transverse shear stress is more significant. Therefore, the present formulation becomes straightforward, whereas self-equilibrating stress fields are not required herein.

Conclusions

The hybrid model is adopted partially in the present model. Therefore, acceptable through thickness transverse shear stress variations are obtained. As the through thickness effect is closely predicted, excellent accuracy and fast convergence are observed for the cross-ply laminates. The computational cost is enormously reduced because of the finite strip method, and only transverse shear stress parameters are required.

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Approximate Vibrational Analysis of Noncircular Cylinders Having Varying Thickness

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Introduction

NONCIRCULAR cylindrical shells have been used in many industrial applications like flight and submarine structures. However, they have received little attention in the area of vibration and stability. The majority of the shell literature published is concerned with circular cylinders, and only a few deal with noncircular cylinders.[1] The variable curvature in noncircular cylinders introduces variable coefficients into their governing differential equations and causes their solution to be difficult. Hence most of the closed-form solutions presented in the literature use simplifying assumptions.

The purpose of the present study is to present an approximate free vibrational analysis for noncircular cylindrical shells having variable thickness along the circumference. In the present work the finite strip method is used since this method makes use of the regular geometry in the longitudinal direction and thereby reduces the size of the problem. A combination of polynomial and harmonic functions that satisfy the boundary conditions is used to represent the displacement components. Numerical results are presented for several oval cylindrical shells, and good comparisons are obtained with those available in the literature.

Analysis

The present analysis is applicable to homogenous, isotropic, elastic, thin-walled, cylindrical shells. The coordinate system and shell geometry are shown in Fig. 1. The bounding surfaces of the shell are assumed to be at distances $z=\pm h/2$ from the middle surface, where h is the wall thickness of the shell. The thickness of the shell at any point along the circumference is given by $h=h_0H(s)$, where h_0 is the thickness at $s=0$ and $H(s)$ is a prescribed function of s. Consistent with the assumptions of thin-shell theory, the thickness is considered to be small in comparison with the other characteristic dimensions of the middle surface.

The kinematic relations used in the present study are based on the Kirchhoff-Love assumptions of classical thin-shell theory, and the final equation of motion is derived with the aid of strain-displacement relations obtained by the simplification of the exact relations developed by Flügge.[2] The axial, circumferential, and radial displacement components of the middle surface are denoted by u^*, v^*, and w^*, respectively (Fig. 1). The displacements (u^*, v^*, and w^*), the axial and circumferential coordinates (x^* and s^*), and the radius of curvature (r^*) are cast into a nondimensionalized form given by $(u,\ v,\ w)=(u^*,\ v^*,\ w^*)/h_0$ and $(x,s,r)=(x^*,\ s^*,\ r^*)/r_0$. The quantity r_0 is defined as the average radius of the noncircular cylindrical shell and is given by $L_s/2\pi$, where L_s is the circumference of the middle surface. The simplified strain-displacement relations, in terms of the nondimensionalized parameters, are given by

$$\epsilon_x = u_{,x} - zw_{,xx}$$
$$\epsilon_s = v_{,s} + (w/r) - zw_{,ss} \qquad (1)$$
$$\epsilon_{xs} = u_{,s} + v_{,x} - 2zw_{,xs}$$

where subscripted commas denote partial differentiations.

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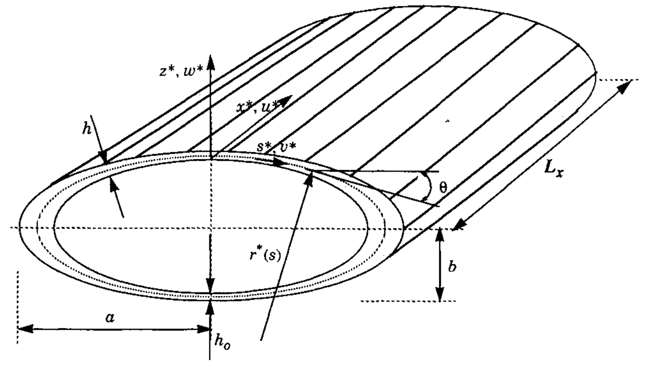

Fig. 1 Shell geometry and coordinate sign convention.

Using the relations just given, Hooke's law for plane stress, and integrating over the shell thickness, the strain energy U and kinetic energy T are expressed in terms of dimensionless middle surface displacements as follows:

$$U = \left(\frac{1}{2}\right) D \int_x \int_s h \Bigg\{ \bigg[u_{,s}^2 + v_{,s}^2 + (w/r)^2 + 2v_{,s}(w/r) + \left(\frac{1-\mu}{2}\right)(u_{,s}^2 + v_{,x}^2 + 2u_{,s}v_{,x}) + 2\mu\{u_{,x}v_{,s} + u_{,x}(w/r)\} \bigg] + k\bigg[w_{,xx}^2 + w_{,ss}^2 + \left(\frac{1-\mu}{2}\right) w_{,xs}^2 + 2\mu w_{,xx}w_{,ss} \bigg] \Bigg\} \, \mathrm{d}x \, \mathrm{d}s \tag{2}$$

$$T = \left(\frac{1}{2}\right) \alpha \int_x \int_s h(\dot{u}^2 + \dot{v}^2 + \dot{w}^2) \, \mathrm{d}x \, \mathrm{d}s \tag{3}$$

In Eqs. (2) and (3),

$$D = (Eh_0^2)/[(1-\mu^2)r_0^2], \quad k = (1/12)(h/r_0)^2, \quad \alpha = \rho\omega^2 h_0^2 \tag{4}$$

where E is the modulus of elasticity, ρ the mass density, and μ the Poisson's ratio.

In the present study, the dimensionless displacements u, v, and w are taken as

$$\begin{aligned} u(x,s,t) &= \sum_{m=1}^{k} [f_1(s)u_1^m + f_2(s)u_{1,s}^m + f_3(s)u_2^m + f_4(s)u_{2,s}^m]\bar{X}_m \cos\omega_m t \\ v(x,s,t) &= \sum_{m=1}^{k} [f_1(s)v_1^m + f_2(s)v_{1,s}^m + f_3(s)v_2^m + f_4(s)v_{2,s}^m]X_m \cos\omega_m t \\ w(x,s,t) &= \sum_{m=1}^{k} [f_1(s)w_1^m + f_2(s)w_{1,s}^m + f_3(s)w_2^m + f_4(s)w_{2,s}^m]X_m \cos\omega_m t \end{aligned} \tag{5}$$

where $f_1(s), \ldots, f_4(s)$ are cubic shape functions of s, m is the longitudinal half-wave number, $\bar{X}_m$ and X_m are harmonic functions that satisfy the boundary conditions at both ends of the cylindrical shell, and ω_m is the circular frequency of a noncircular cylindrical shell corresponding to the longitudinal mode m. The subscripts 1; 2; 1,s; and 2,s on displacements u^m, v^m, and w^m denote corresponding displacements and their derivatives at nodal lines 1 and 2, respectively, of a finite strip for mth vibrational mode. For the case of a cylinder with freely supported edges ($u_{,x} = 0, v = 0, w = 0$, and $w_{,xx} = 0$ at edges $x^* = 0$ and L_x), the harmonic functions used are $X_m = \sin(m\pi r_0 x/L_x)$ and $\bar{X}_m = \cos(m\pi r_0 x/L_x)$. For other boundary conditions, harmonic functions are given in Ref. 3. Substituting Eq. (5) into Eqs. (2) and (3), the total energy $\Pi = U - T$ can be expressed in terms of nodal displacements. Applying the Hamilton's variational principle to the total energy functional Π, the following equation of motion is obtained:

$$\sum_{m=1}^{k} ([K]_m\{\Delta\}_m - \Omega_m^2[M]_m\{\Delta\}_m) = \{0\} \tag{6}$$

where $\{\Delta\}_m$ is the nodal line displacement vector corresponding to the mth vibrating mode that is shown as follows:

$$\{\Delta\}_m^T = [u_1^m \ u_{1,s}^m \ v_1^m \ v_{1,s}^m \ w_1^m \ w_{1,s}^m \ u_2^m \ u_{2,s}^m \ v_2^m \ v_{2,s}^m \ w_2^m \ w_{2,s}^m] \tag{7}$$

and

$$\Omega_m = \omega_m r_0 \sqrt{\rho(1-\mu^2)/E} \tag{8}$$

The parameter Ω_m is the nondimensional frequency of a noncircular cylindrical shell corresponding to the longitudinal mode m.

Representation of Varying Radius of Curvature

Since the solution of Eq. (6) strongly depends on the variable curvature, it follows that the radius of curvature should represent the actual geometry of the middle surface as close as possible. In the present analysis, the radius of curvature is taken to be a simplified, one-parameter version of Marguerre's[4] Fourier series representation of the cylinder curvature. This curvature expression, which represents a doubly symmetric oval (and approximates an ellipse having the same major-to-minor axis ratio as the oval) is given as

$$r_0/r = 1 + \xi \cos(2s) \tag{9}$$

where $0 \le s \le 2\pi$, and $-1 \le \xi \le 1$. To prevent a re-entrant cross section (negative curvature), the absolute value of ξ must not be greater than 1, which corresponds to a major-to-minor axis ratio of $a/b \le 2.06$. For a given major-to-minor axis ratio, the parameter ξ is very closely approximated by[4]

$$\xi = 3Q - (36/35)Q^3 \tag{10}$$

where $Q = [(a/b - 1)/(a/b + 1)]$.

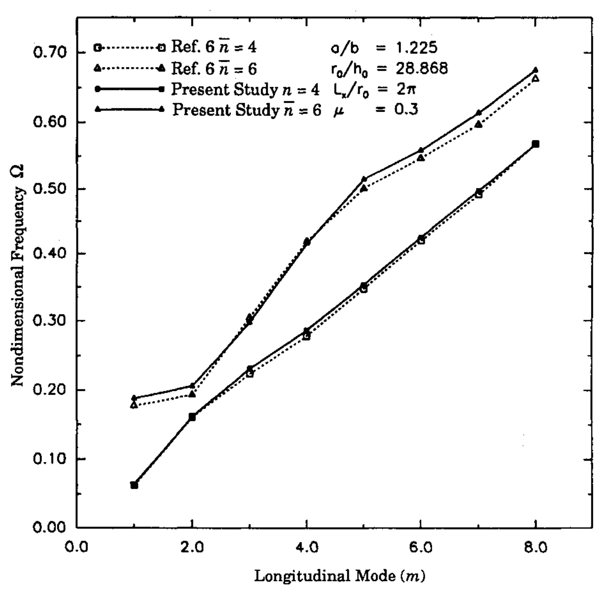

Fig. 2 Comparison of nondimensional frequencies of freely supported elliptical shells having parabolic thickness variation along the circumference (symmetric, $\xi = 0.30416$, $\epsilon = +0.2$).

Table 1 Comparison of nondimensional frequencies (Ω_1) for symmetric and antisymmetric cases [$L_x/r_o = 1.0$, $r_o/h_o = 91.7$, $m = 1$, $H(s) = 1$]

	Symmetric modes				Antisymmetric modes			
	$\xi = 0.5$		$\xi = -0.5$		$\xi = 0.5$		$\xi = -0.5$	
$\bar{n}$	Ref. 5	Present study	Ref. 5	Present study	Ref. 5	Present study	Ref. 5	Present study
1	1.07067	1.07070	0.74000	0.73990	0.74000	0.73978	1.07067	1.07070
2	0.54898	0.54910	0.54898	0.54910	0.69257	0.69268	0.69257	0.69268
3	0.48080	0.48100	0.42624	0.42642	0.42624	0.42642	0.48080	0.48100
4	0.33789	0.33818	0.33789	0.33818	0.35395	0.35443	0.35395	0.35492
5	0.28024	0.28055	0.27569	0.27598	0.27569	0.27598	0.28024	0.28055
6	0.20638	0.20658	0.20638	0.20658	0.20810	0.20830	0.20810	0.20830
7	0.20817	0.20837	0.20655	0.20674	0.20655	0.20674	0.20817	0.20838
8	0.27003	0.27039	0.27003	0.27039	0.27152	0.27189	0.27152	0.27189
9	0.30965	0.31000	0.30899	0.30940	0.30899	0.30940	0.30965	0.31000
10	0.35630	0.35665	0.35630	0.35665	0.35735	0.35768	0.35735	0.35759

Numerical Results and Discussions

A computer program based on the analysis described herein has been developed to study the vibrations of noncircular cylindrical shells with thickness varying along the circumference. Results for uniform thickness, circular cylindrical shells [$\xi = 0$ and $H(s) = 1$] are obtained, and good agreement with known results has been established.

Computations are made for oval shells with $L_x/r_0 = 1.0$, $r_0/h = 91.7$, $\mu = 0.3$, $H(s) = 1$, and $m = 1$ and curvature eccentricities $\xi = +0.5$ and -0.5. Since the oval is doubly symmetric, only a quarter section is analyzed by introducing the following boundary conditions at $s = 0$ and $s = \pi/2$.

1) Even symmetric modes (symmetric about both axes) at $s = 0$ and $\pi/2$:

$$u_{,s} = v = w_{,s} = 0$$

2) Odd symmetric modes (antisymmetric about axis passing through $s = \pi/2$, $3\pi/2$) at $s = 0$:

$$u_{,s} = v = w_{,s} = 0$$

at $s = \pi/2$:

$$u = v_{,s} = w = 0$$

3) Even antisymmetric modes (antisymmetric about both axes) at $s = 0$ and $\pi/2$:

$$u = v_{,s} = w = 0$$

4) Odd antisymmetric modes (symmetric about axis passing through $s = \pi/2$, $3\pi/2$) at $s = 0$:

$$u = v_{,s} = w = 0$$

at $s = (\pi/2)$: (11)

$$u_{,s} = v = w_{,s} = 0$$

The solution for one set of shell parameters (r_0, h_0, L_x, m, μ, and ξ) separates into four independent subsolutions, one for each possible combination of symmetric or antisymmetric displacements and even or odd circumferential mode numbers.

Because of the presence of coupling among the circumferential modes, the conventional method of describing the mode by the number of half-waves along the circumference is inadequate. In the present study a parameter $\bar{n}$, called the nominal circumferential wave number (as introduced in Ref. 5), has been used. This parameter $\bar{n}$ is equal to one-half of the number of nodes along the circumference when the curvature eccentricity parameter ξ is equal to zero. The results obtained by the present analysis and by Culberson and Boyd[5] are given in Table 1. Complete agreement with the results obtained by Culberson and Boyd[5] can be seen.

Results for noncircular cylindrical shells with variable thickness were also obtained using the analysis presented herein. The geometry of the noncircular shell analyzed is similar to the elliptical shell used in Ref. 6. The shell parameters used in defining the problem are $a/b = 1.225$, $r_0/h_0 = 28.868$, $L_x/r_0 = 2\pi$, $\mu = 0.3$, and $\xi = 0.304128$. A second degree thickness variation in each quadrant is assumed (see Fig. 1). The thickness is specified in terms of θ only to maintain similarity with the results presented in Ref. 6. The thickness function H is given as

$$H(\theta) = 1 + \epsilon\theta^2 \tag{12}$$

where the parameter ϵ is an arbitrary constant and θ is the angle between the tangent at the origin of s and the one at any point on the centerline. Calculations have been done for three cases of ϵ corresponding to $+0.2$, 0.0, and -0.2, which correspond to ratios of thickness at $s = \pi/2$ to that at $s = 0$ of 1.493, 1, and 0.507, respectively. The cross section is doubly symmetric about both major and minor axes.

The nondimensional frequencies Ω obtained using the present analysis were compared with corresponding results presented in Ref. 6. As seen in Fig. 2, the frequencies obtained by the present analysis compare well with those presented in Ref. 6. The calculated frequencies deviate from $+0.1$ to -5.7% from those given in Ref. 6. These differences in results can be attributed to the fact that Suzuki and Leissa[6] used an elliptical cylinder profile whereas an oval cylinder profile is used in the present study.

Conclusions

An approximate analysis for studying the free vibration of a noncircular cylindrical shell having circumferentially varying thickness is presented. The analysis is formulated to overcome the mathematical difficulties associated with mode coupling caused by variable shell wall curvature and thickness. Accurate results have been obtained for symmetric and antisymmetric vibration modes of shells with oval cross sections. This analysis can be applied to any noncircular cylinder if the curvature along the circumference can be expressed as a function of its perimetrical length. Similarly, any general thickness variation that can be described by the perimetrical length (or the angle θ) can also be analyzed. Thus, the analysis can be applied to nonsymmetric cylindrical shells with both closed and open profiles.

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